HYPEREUCLIDEAN MANIFOLDS
AND THE NOVIKOV CONJECTURE

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ABSTRACT. We develop some basic Lipschitz homotopy technique and apply it to manifolds with finite asymptotic dimension. In particular we show that the Higson compactification of a uniformly contractible manifold is mod p acyclic in the finite dimensional case. Then we give an alternative proof of the Higher Signature Novikov Conjecture for the groups with finite asymptotic dimension. Finally we define an asymptotically piecewise Euclidean metric space as a space which admits an approximation by Euclidean asymptotic polyhedra. We show that the Gromov-Lawson conjecture holds for the asymptotically piecewise Euclidean groups. Also we prove that expanders are not asymptotically piecewise Euclidean

§1 INTRODUCTION

The Novikov Conjecture states that the higher signatures of a manifold are homotopy invariants. The ordinary signature of a manifold M can be computed by the Hirzebruch formula $\sigma(M) = \langle L, [M] \rangle$ where the right hand side is the integration of the Hirzebruch polynomial over the fundamental homology class of M. If the fundamental group $\Gamma = \pi_1(M) \neq 0$, then there are distinguished homology classes $\beta \in H_*(M)$ which come from the group Γ. The integration of the Hirzebruch polynomial over these classes β gives rise the higher signatures $\sigma_\beta(M)$ of M. In view of this it makes sense to speak about the Novikov Conjecture for a given finitely presented group Γ.

The most successful approach to the Novikov Conjecture in the last decade was the so called coarse approach. The coarse approach consists of studying the large scale geometry of a group Γ as a metric space with the word metric. Quite often it is more convenient to consider the universal cover $\tilde{M}$ with the induced metric, though in the coarse sense $\tilde{M}$ is

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equivalent to Γ. The most advanced result on the Novikov Conjecture is a theorem of G. Yu [Yu2] which asserts that the Novikov Conjecture holds true for groups Γ that admit an embedding in a coarse sense in the Hilbert space l_2. This Yu's theorem formally generalizes the other his theorem [Yu1] saying that the Novikov Conjecture holds for groups Γ with finite asymptotic dimension $\operatorname{asdim} \Gamma$. Both Yu's result are obtained by proving the coarse Baum-Connes conjecture which states that the Roe index map

$$A : K_*(\tilde{M}) \to K_*(C^*(\tilde{M}))$$

is an isomorphism when $\tilde{M}$ is the universal cover of a closed aspherical manifold M [H-R],[Ro2]. The C^*-algebra $C^*(\tilde{M})$ here is the completion of the algebra of infinite matrices $A = (a_{x,y})_{x,y \in \Gamma}$ whose entries are compact operators on l_2 and $a_{x,y} = 0$ for $d_\Gamma(x,y) \geq r$ for some $r = r(A)$. Thus both Yu's works are heavily operator algebra theoretic. The descent from the coarse Baum-Connes conjecture to the Novikov Conjecture is shown in [Ro2]. It uses the homotopy fixed point theory and it proves the analytic Novikov conjecture. The analytic Novikov conjecture states that the analytic assembly map $A_\Gamma :$ $K_*(B\Gamma) \to K_*(C_r^*\Gamma)$ is injective. Here $C_r^*\Gamma$ is the reduced group C^*-algebra. The classical Novikov Conjecture is equivalent to the statement that the rational assembly map from the surgery exact sequence $H_*(B\Gamma; \mathbf{L}) \otimes \mathbf{Q} \to L_*(\mathbf{Z}[\Gamma]) \otimes \mathbf{Q}$ is injective [K-M], [F-R-R]. It is known that the analytic Novikov conjecture implies the original conjecture [F-R-R].

Despite on recent counterexamples to the coarse Baum-Connes conjecture, this approach to the Novikov Conjecture is not exhausted yet. It is possible to show that the equivariant split injectivity of the Roe index map also enables to derive the Novikov Conjecture. As far as it known to the author, the monomorphism version of the coarse Baum-Connes conjecture is not disproved to this moment.

A more geometric coarse reduction of the Novikov Conjecture was considered by Ferry and Weinberger [F-W]. They were looking for obstructions to make a homotopy between two closed aspherical manifold tangential. This approach can be traced back to Farrell-Hsiang [F-H]. The Ferry-Weinberger approach reduces the Novikov Conjecture to the question whether the boundary homomorphism $\delta : \check{H}^{n-1}(\nu \tilde{M}; \mathbf{Q}) \to H_c^n(\tilde{M}; \mathbf{Q})$ is an equivariant split epimorphism where $\nu \tilde{M}$ is the Higson corona [Ro1].

We recall that the Higson compactification $X \cup \nu X$ of X is generated by the algebra of functions with the gradient tending to zero at infinity. The exact sequence of pair $(\tilde{M} \cup \nu \tilde{M}, \nu \tilde{M})$ implies that the boundary homomorphism δ is an epimorphism provided $H^n(\tilde{M} \cup \nu \tilde{M}; \mathbf{Q}) = 0$. The assertion that the Higson compactification $\tilde{M} \cup \nu \tilde{M}$ is acyclic is called the Higson conjecture [Ro1]. It is known that for integral coefficients this conjecture false even for $\mathbf{R}^n$ [Ke],[D-F]. In §3 we prove the Higson conjecture for finite

coefficients in the finite dimensional case. There are chances that the mod p Higson conjecture holds in full generality. In §5 we formulate a stable version of the Higson conjecture which cannot be disproved by simple examples.

The other, in some sense equivalent, approach to the Novikov Conjecture is due to Gromov which goes back to his work on the positive scalar curvature [G-L]. He introduced the notions of hyperspherical and hypereuclidean manifolds [G2]. Roe proved [Ro1] that an n-manifold $\tilde{M}$ is hypereuclidean if and only if the Higson corona $\nu\tilde{M}$ admits a map of degree one onto an $n - 1$-sphere S^{n-1}. It is easy to show that if a group Γ acts on $\tilde{M}$ by isometries then that action can be extended to an action on the Higson corona [Dr1]. We introduce the notion of an equivariantly hypereuclidean (or Γ-hypereuclidean) n-manifold as a manifold $\tilde{M}$ with a group Γ acting on it properly and cocompactly whose Higson corona admits a map f of degree one onto an $(n - 1)$-dimensional sphere such that the action of Γ is fixed on $f^*(H^{n-1}(S^{n-1}))$. Using Ferry-Weinberger approach it is possible to show that the Novikov Conjecture holds for manifolds M with the equivariantly $\pi_1(M)$-hypereuclidean universal cover $\tilde{M}$. In §4, §5 we show that if $\operatorname{asdim}\pi_1(M) < \infty$ then $\tilde{M} \times \mathbf{R}^n$ is equivariantly hypereuclidean for some n. This allows to establish the Novikov Conjecture for M.

The concepts of hypersphericity is formally weaker. There is an open question whether an integrally hyperspherical manifold is always hypereuclidean. The hypersphericity of $\tilde{M}$ implies the Gromov-Lawson conjecture for M [G-L]: *A closed aspherical manifold cannot carry a metric of a positive scalar curvature.* This conjecture is quite close to the Novikov Conjecture [Ros]. Perhaps the Novikov Conjecture can be derived from some equivariant version of the hypersphericity. In [Dr2] it was shown that in finite dimensional case the manifold $\tilde{M} \times \mathbf{R}^k$ is hyperspherical for some k.

In this paper in §6 we introduce the notion of *asymptotically piecewise Euclidean metric space* in the coarse category as a space that admits an approximation by piecewise Euclidean asymptotic polyhedra. Then we extend Yu's result about Gromov-Lawson conjecture to asymptotically piecewise Euclidean manifolds $\tilde{M}$. Note that all groups uniformly embeddable in the Hilbert space l_2 are asymptotically piecewise Euclidean, although it is unclear whether the inclusion of these classes is strong. Like in the case with Yu's theorem [G4], this result is also limited by expanders. This is demonstrated in §7, where we show that an asymptotically piecewise Euclidean metric space cannot contain an expander.

We note that there is a connection between Ferry-Weinberger's (and Gromov's) and the coarse Baum-Connes conjecture approaches to the Novikov Conjecture which is based on the fact that the topological K-theory $K_*(\nu\tilde{M})$ is an approximation to C^*-algebra K-theory $K_*(C^*(\tilde{M}))$ [Ro1].

Using Roe's coarse cohomology one can define an asymptotic cohomological dimension asdim$_\mathbf{Z}$ X in the coarse category as it was done in [Dr1]. It seems to do that properly one has to make a shift in the grading of Roe's cohomology (it was not done in [Dr1]). Only in that case we would get the equality asdim$_\mathbf{Z}$ $\mathbf{R}^n = n$. The Roe cohomology is dual in the macro-micro sense to the Steenrod homology. Curiously, Steenrod defined his homology also with shifted dimensions [St] and only later Sitnikov made the correction [Sit]. We note that for the universal covers $\tilde{M}$ of aspherical manifolds always there is the inequality asdim$_\mathbf{Z}$ $\tilde{M} < \infty$ [Dr1]. If the asymptotic cohomological dimension agreed with Gromov's asymptotic dimension, then the Novikov Conjecture would follow. In the ordinary topology the problem about coincidence of the cohomological dimension and the Lebesgue dimension was known from late 20s as the Alexandroff problem. The Alexandroff problem was solved by a counterexample [Dr3]. In the large scale world a counterexample was constructed in [D-F-W] but that example does not have a bounded geometry. Recently a counterexample among finitely presented groups was found by Gromov by means of expanders and random groups [G].

§2 Lipschitz homotopy

In this section we consider variations of the following question: *When does a null homotopic λ-Lipschitz map $f : X \to Y$ admit a μ-Lipschitz homotopy $H : X \times I \to Y$ to a constant map?* We recall that a map $f : X \to Y$ between metric spaces is λ-Lipschitz if $d_Y(f(x), f(x')) \le \lambda d_X(x, x')$ for all $x, x' \in X$. Denote by $L(f) = \sup\{\frac{d_Y(f(x), f(x'))}{d_X(x, x')}\}$. Then for a λ-Lipschitz map f we have $L(f) \le \lambda$.

First we give an answer to the question for finite simplicial complexes. Every simplicial complex K carries a metric where all simplexes are the standard of size one. We will call such metric *uniform* and usually we will denote corresponding metric space as K_U. If it is not specified, we will assume that a finite complex always supplied with the uniform metric.

Lemma 2.1. *Suppose that X and Y are finite simplicial complexes. Then for every λ there exists $\mu = \mu(\lambda)$ such that every null homotopic λ-Lipschitz map $f : X \to Y$ admits a μ-Lipschitz homotopy $H : X \times I \to Y$ to a constant map.*

Proof. Let r_n denote radius of the inscribed sphere in the standard n-simplex. We fix a subdivision T of X with the mesh $< \frac{r_n}{4\lambda}$. There are finitely many different simplicial maps $\phi : T \to Y$. We consider only null homotopic maps ϕ_i. For every i we fix a homotopy $H_i : T \times I \to Y$ to a constant map. By Simplicial Approximation Theorem we

may assume that the map H_i is μ_i-Lipschitz. We take $\mu \geq \max\{2\mu_i\}$. Also we assume that $\mu \geq \frac{1}{2r_T}$ where r_T is the minimum of radii of inscribed spheres in simplices of T. According to the following Lemma 2.2 every λ-Lipschitz map $f : X \to Y$ is homotopic to a simplicial map $g : T \to Y$ with respect to T. Moreover, the corresponding homotopy is $(\mu/2)$-Lipschitz with the above μ. Then $g = \phi_i$ for some i and a μ-Lipschitz homotopy of f to a constant map will be the sum of the above homotopy deforming f to g and H_i. $\square$

By $B_r(x_0) = \{x \in X \mid d(x, x_0) \leq r\}$ we denote a closed ball of radius r in a metric space X with center at $x_0 \in X$. The unit n-ball in $\mathbf{R}^n$ will be denoted as B^n.

Lemma 2.2. *Let $f : L \to K$ be a λ-Lipschitz map between uniform finite dimensional polyhedra. Then f is homotopic to a simplicial map g with respect to some subdivision of L by means of μ-Lipschitz homotopy, where μ depends on λ and $\dim L, \dim K$ only.*

Proof. We apply the standard argument of the Simplicial Approximation theorem. Consider an open cover $\mathcal{U} = \{f^{-1}(OSt(v, K)) \mid v \in K^{(0)}\}$ of L where $OSt(v, K)$ denotes the open star of a vertex v in a complex K. Since f is λ-Lipschitz, the Lebesgue number of $\mathcal{U}$ is greater than $\frac{r_n}{2\lambda}$, where r_n denotes radius of the inscribed sphere in the unit standard n-simplex and $n = \dim K$. To see that, take a point $y \in L$ and consider a closest vertex $v \in K^{(0)}$ to $f(y)$. Then the ball $B_{\frac{r_n}{2}}(y)$ is contained in $OSt(v, K)$. Hence the ball $B_{\frac{r_n}{2\lambda}}$ is contained in $f^{-1}(OSt(v, K))$. Consider a triangulation T of L which is an iterated barycentric subdivision of K with mesh $< \frac{r_n}{4\lambda}$. Let r_T denote the minimum of radii of inscribed spheres in simplices of T. We take $\mu \geq \frac{1}{r_T}$. Then μ depends on λ, n and dimension of L. We consider a simplicial approximation $g : T \to K$ defined by the standard rule: for every vertex $a \in T^{(0)}$ we take $g(a) \in K^{(0)}$ such that $OSt(a, L) \subset f^{-1}(OSt(g(a), K))$. Let us show that for every $x \in L$ the image $g(x)$ belongs to σ whenever $f(x)$ lies in the interior of a simplex σ in K. It suffices to show that $g(\Delta) = \sigma$, where Δ is a unique simplex in L containing x as an interior point. Let $p \in \Delta^{(0)}$ be a vertex, then $x \in OSt(p, L)$. Therefore $f(x) \in f(OSt(p, L)) \subset OSt(g(p), K)$. This means that $g(p)$ is a vertex of σ. Then a Lipschitz homotopy between f and g is defined linearly by joining $f(x)$ with $g(x)$ in σ. $\square$

REMARK. Lemma 2.1 holds true if one consider a compact metric space X. In that case by the Ascoli-Arzela theorem the space Y_λ^X of λ-Lipschitz maps $f : X \to Y$ is compact. We can take a finite ϵ-net in Y_λ^X for small enough ϵ. Similarly, for every map $\phi_i : X \to Y$ from this ϵ-net we fix a Lipschitz homotopy H_i to a constant map. Then any other λ-Lipschitz map is μ'-homotopic to one from the net.

The following lemma can be derived also from results of Siegel and Williams [S-W].

Lemma 2.3. *Let Y be a finite simplicial complex with $\pi_n(Y)$ finite. Then for every λ there is μ such that every map $f : B^n \to Y$ with $L(f|_{S^{n-1}}) \le \lambda$ can be deformed to a μ-Lipschitz map $g : B^n \to Y$ by means of a homotopy $h_t : B^n \to Y$ with $h_t|_{S^{n-1}} = f|_{S^{n-1}}$.*

Proof. We consider a finite family of simplicial maps $\phi_i : S^{n-1} \to Y$ as in the proof of Lemma 2.1. Since the group $\pi_n(Y)$ is finite, there are finite number of homotopically different extensions $H_i^j : B^n \to Y$. We assume that every H_i^j is μ_i^j-Lipschitz. The rest of the proof is the same as in Lemma 1. $\square$

Lemma 2.4. *Let L be a finite dimensional complex and let K be a finite complex with finite homotopy groups $\pi_i(K)$ for $i \le \dim L + 1$. Let $f, g : L \to K$ be homotopic Lipschitz maps. Then every homotopy between f and g can be deformed to a Lipschitz homotopy $H : L \times [0,1] \to K$.*

Proof. Let $F : L \times I \to K$ be a homotopy between f and g. By induction on n, and using Lemma 2.3 we construct a μ_n-Lipschitz map $H_n : L^{(n)} \times I \cup L \times \{0,1\} \to K$ which is a deformation of F restricted to the n-skeleton $L^{(n)}$ such that H_n extends H_{n-1}. Then $H = H_m$ for $m = \dim L$. $\square$

§3 Modulo p Higson conjecture

A countable simplicial complex L with a geodesic metric on it is called an *asymptotic polyhedron* if every simplex in L is isometric to a simplex Δ spanned in the Hilbert space and the radii of inscribed spheres r_Δ tend to infinity. In this definition any Banach space can be used. Moreover, asymptotic polyhedra naturally appear with l_∞ metric. Since we are working in this section with finite dimensional complexes we can consider only the Euclidean case.

We recall that by the definition the *asymptotic dimension* $\operatorname{asdim} X$ of a metric space X does not exceed n if for any arbitrary large number d there is a uniformly bounded open cover $\mathcal{U}$ of X with multiplicity $\le n + 1$ and with the Lebesgue number $\ge d$ [G1]. This is equivalent that for arbitrary small λ a space X admits a uniformly cobounded λ-Lipschitz map onto an n-dimensional uniform simplicial complex [G1].

We note that for every n-dimensional asymptotic polyhedron L, $\operatorname{asdim} L = n$.

Lemma 3.1. *Suppose that $\operatorname{asdim} X \le n$ and $f : X \to \mathbf{R}_+$ is a given proper function. Then there are a compact set $C \subset X$ and a 1-Lipschitz map $\phi : X \to L$ to an n-dimensional asymptotic polyhedron with $\operatorname{diam}(\phi^{-1}(\Delta)) \le f(z)$ for all $z \in \phi^{-1}(\Delta) \setminus C$. Moreover L can be presented as the union $L_0 \cup J_0 \cup L_1 \cup J_1 \cup \ldots$, where each L_i is a uniform polyhedron of size 2^i and edges in each complex J_i have the length 2^i or 2^{i+1}.*

Proof. Let $\mathcal{U}$ be an open cover of X of multiplicity $\leq n+1$ with the Lebesgue number $> 2\lambda$ and with $\max\{\operatorname{diam} U \mid U \in \mathcal{U}\} \leq R$. We may assume that $\max_{x \in X} d(x, X \setminus U) > 2\lambda$ for all $U \in \mathcal{U}$. Denote $\phi(x) = d(x, X \setminus U)$. The family of maps $\{\phi_U\}_{U \in \mathcal{U}}$ defines an 1-Lipschitz map $\phi' : X \to l_\infty$ to the Banach space l_∞. Note that $\phi'(X) \subset l_\infty \setminus B_{2\lambda}(0)$. Also we note that the radial projection π of $l_\infty \setminus B_{2\lambda}(0)$ onto the sphere $S_\lambda(0)$ is 1-Lipschitz. Then the map $\phi = \pi \circ \phi' : X \to S_\lambda(0)$ is 1-Lipschitz. Since the multiplicity of the cover $\mathcal{U}$ does not exceed $n + 1$, the image $\phi(X)$ lies in the n-skeleton of the spherical infinite dimensional simplex $\Delta^\infty = S_\lambda(0) \cap l_\infty^+$ of the size λ. In other words the nerve $N(\mathcal{U})$ of the cover $\mathcal{U}$ is contained as a subcomplex in the n-skeleton of Δ^∞. There is a constant c_n which depends on n only such that the identification of an n-simplex in Δ^∞ with the standard Euclidean n-simplex of the size $c_n\lambda$ is 1-Lipschitz. Now we have that the map $\phi : X \to N(\mathcal{U})$ is 1-Lipschitz, where the nerve $N(\mathcal{U})$ is given a uniform metric of size $c_n\lambda$. Note that $\phi^{-1}(\Delta) \subset \overline{St(x, \mathcal{U})} \subset B_R(x)$ for some $x \in \phi^{-1}(\Delta)$ for every simplex Δ.

We consider a sequence of such covers $\mathcal{U}_k$ with $\lambda_k > \frac{1}{c_n} 2^k$; $k = 0, 1, \ldots$ and define $C = f^{-1}([0, 2R_2])$. Then we construct compact sets $C_k \supset f^{-1}([0, 2R_{k+3}])$ such that $d(C_k \setminus C_{k-1}, C_{k-2}) \geq R_{k+2}$. Let $\tilde{\mathcal{U}}_k = \{U \in \mathcal{U}_k \mid U \cap (C_k \setminus C_{k-1}) \neq \emptyset\}$. We form a cover $\tilde{\mathcal{U}} = \cup_k \tilde{\mathcal{U}}_k$. Then the nerve $N(\tilde{\mathcal{U}})$ has a required type $L_0 \cup J_0 \cup L_1 \cup J_1 \cup \ldots$. It can be given the metric where each L_i is a uniform complex with the size of simplices 2^i and every J_i is a complex with simplices of a mixed type: their edges which formed by two elements of the cover $\tilde{\mathcal{U}}_i$ have the length 2^i, and the edges which are formed by two elements of $\tilde{\mathcal{U}}_{i+1}$ or by one element from $\tilde{\mathcal{U}}_i$ and the other element from $\tilde{\mathcal{U}}_{i+1}$ have the length 2^{i+1}. Here L_i is the nerve of the cover $\tilde{\mathcal{U}}_k$ and J_i is the nerve of the restriction of $\mathcal{U}_k \cup \mathcal{U}_{k+1}$ over the boundary ∂C_k. If $x \in C_k \setminus C_{k-1}$, then $f(x) \geq 2R_{k+1}$. Note that $\phi^{-1}(\Delta) \subset B_{R_{k+1}}(x)$ for some $x \in \phi^{-1}(\Delta)$. Then for any $z \in \phi^{-1}(\Delta)$ we have $d(z, x) \leq R_{k+1}$. Hence $z \notin C_{k-2}$ and therefore $f(z) \geq 2R_{k+1} \geq diam(\phi^{-1}(\Delta))$.

The complex $N(\tilde{\mathcal{U}})$ satisfies all the requirements except it is not necessarily n-dimensional. The complexes L_i are n-dimensional the best estimate for dimension of complexes J_i is $2n + 1$. The n-dimensionality can be achieved by some standard dimension theoretic trick with the choice of covers. We are not giving all details, since for the purpose of this paper the finite dimensionality is enough. $\square$

A metric space X is called *uniformly contractible* if there is a function $S : \mathbf{R}_+ \to \mathbf{R}_+$ such that every ball $B_r(x)$ is contractible to a point in the ball $B_{S(r)}(x)$.

Let $x_0 \in X$ be a base point. We denote $\|x\| = d_X(x, x_0)$.

Lemma 3.2. *Let X be a uniformly contractible proper metric space with* $\operatorname{asdim} X = n$. *Then given a proper function $g : X \to \mathbf{R}_+$ there are an n-dimensional asymptotic polyhedron N, a proper 1-Lipschitz map $\phi : X \to N$, and a proper homotopy inverse map*

$\gamma : N \to X$ with $d(x, \gamma\phi(x)) < g(x)$ for all $x \in X$.

Proof. We define by induction on i a lift γ on the i-skeleton $N^{(i)}$ of the nerve of a cover of X given by Lemma 3.1 for an appropriate choice of f. We take $\gamma(v) \in \phi^{-1}(v)$ for every vertex v. Then using the uniform contractibility of X we can extend γ with control over the 1-skeleton $N^{(1)}$ and so on. Without loss of generality we may assume that X is a polyhedron of the dimension n supplied with a triangulation of mesh ≤ 1. By induction on i we define a homotopy $H : X^{(i)} \times I \to X$ joining the identity map with $\gamma \circ \phi$. We consider a function $\psi(x) = \|x\| - \max\{d(x, y) \mid y \in H(x \times I)\}$. If ψ tends to infinity, then the map H is proper. Therefore it suffices to show that ψ tends to infinity for an appropriate choice of f. Let S be a contractibility function. We define $\rho(t) = S^{-1}(t/2)$ where S^{-1} is the inverse function for S. Then we take $f = \rho^{2n+1} \circ g$, the composition of g and $2n+1$ times iteration of ρ. We assume here that $g(x) \leq \|x\|/2$. Then it is easy to verify that $\psi(x) \geq \|x\|/2$. $\square$

A metric space X is called *proper* if every ball $B_r(x)$ in X is compact. We recall that the Higson compactification $\bar{X}$ of a proper metric space X is defined by the ring of bounded functions with the gradient tending to zero at infinity [Ro1]. The reminder of this compactification is called the *Higson corona* and it is denoted as νX. Thus, $\bar{X} = X \cup \nu X$. The defining property of the Higson corona is the following:

(*) a continuous map $f : X \to Z$ to a compact metric space is extendable over the Higson corona νX if and only if for every R diameter of the image $f(B_R(x))$ of the R-ball centered at x tends to zero as x approaches infinity.

Note that a proper Lipschitz map $f : X \to Y$ induces a continuous mapping between the Higson coronas $\bar{f} : \nu X \to \nu Y$.

Theorem 3.3. *Let X be a uniformly contractible proper metric space with a finite asymptotic dimension and let $\bar{X}$ be the Higson compactification. Then $\check{H}^n(\bar{X}; \mathbf{Z}_p) = 0$ for all n and all p.*

Proof. We show that every map $\alpha : \bar{X} \to K(\mathbf{Z}_p, n)$ is null homotopic. Since $\bar{X}$ is compact, the image $\alpha(\bar{X})$ is contained in the k-skeleton $K = K(\mathbf{Z}_p, n)^{(k)}$, $k > \mathrm{asdim}\, X + 1$, which is a finite complex. We fix a geodesic metric on K. Let ϵ_K be an injectivity radius in K, i.e. every two points within a distance ϵ_K can be joined by a unique geodesic. Since the map $\alpha|_X \to K$ is extendable over the Higson corona the function $R_\alpha(t) = L(\alpha|_{X \setminus B_t(x_0)})$ tends to zero at infinity. We apply Lemma 3.2 with $g(x) \leq \min\{\epsilon_K/R_\alpha(\|x\|/2), \|x\|/4\}$ to obtain an asymptotic polyhedron N and maps $\phi : X \to N$ and $\gamma : N \to X$. Let $[u, v]$ be an edge in N, then $d_K(\alpha\gamma(u), \alpha\gamma(v)) \leq R_\alpha(t_0)d_X(\gamma(u), \gamma(v))$ where $t_0 = \min\{\|\gamma(u)\|, \|\gamma(v)\|\}$. We may assume that there are $x, y \in X$ such that $\phi(x) = u$

and $\phi(y) = v$. Then

$$d_X(\gamma(u), \gamma(v)) \le d_X(x,y) + \epsilon_K/R_\alpha(\tfrac{1}{2}\|x\|) + \epsilon_K/R_\alpha(\tfrac{1}{2}\|y\|) \le diam\phi^{-1}[u,v] + 2\epsilon_K/R_\alpha(\tfrac{1}{2}\|x\|)$$

provided $\|x\| \le \|y\|$. Because of the inequality $g(x) \le \|x\|/4$ we have that $R_\alpha(t_0) \le R_\alpha(\|x\|/2)$. We may assume that $f < g$ and then $diam(\phi^{-1}([u,v])) < g(x)$. Summarizing all this, we obtain the inequality $d_K(\alpha\gamma(u), \alpha\gamma(v)) \le 3\epsilon_K$. This means that the map $\alpha \circ \gamma \circ u^{-1}$ is $3\epsilon_K$-Lipschitz where $u : K \to K_U$ is the projection to the uniform metric.

Since X is contractible, the map $\alpha \circ \gamma$ is null homotopic. Note that the homotopy groups $\pi_i(K)$ are finite for $i \le \dim N + 1$. We apply Lemma 2.4 to obtain a λ-Lipschitz homotopy $H : N_U \times I \to K$ of $\alpha \circ \gamma$ to a constant map. This homotopy defines a Lipschitz map $\tilde{H} : N_U \to K_\lambda^I$ to the space of λ-Lipschitz mappings of the unit interval I to K. Then the composition $\tilde{H} \circ u : N \to K_\lambda^I$ satisfies the Higson extendibility condition (*). Let $\bar{h} : \bar{N} \to K_\lambda^I$ be the extension over the Higson corona. This extension defines a map $\bar{H} : \bar{N} \times I \to K$. The map $\bar{H}$ is a homotopy between the extension $\xi = \overline{\alpha \circ \gamma}$ and a constant map. To complete the proof, we show that α is homotopic to $\xi \circ \bar{\phi}$ where $\bar{\phi}$ is the extension of the Lipschitz map ϕ to the Higson compactifications. Note that

$$d_K(\alpha(x), \alpha\gamma\phi(x)) \le R_\alpha(t_0)d(x, \gamma\phi(x)) \le R_\alpha(t_0)\epsilon_K/R_\alpha(\|x\|/2) \le \epsilon_K$$

where $t_0 = \min\{\|x\|, \|\gamma\phi(x)\|\} \ge \|x\|/2$. Then for every $x \in X$ we join the points $\alpha(x)$ and $\alpha\gamma\phi(x)$ by the unique geodesic $\psi_x : I \to K$. This defines a map $\tilde{\psi} : X \to K_\mu^I$. Since both α and $\alpha \circ \gamma \circ \phi$ satisfy the condition (*), the map $\tilde{\psi}$ has the property (*). Let $\bar{\psi} : \bar{X} \to K_\mu^I$ be the extension of $\tilde{\psi}$ to the Higson corona. The map $\bar{\psi}$ defines a homotopy $\Psi : \bar{X} \times I \to K$ between α an $\xi \circ \bar{\phi}$. $\square$

A potential application of Theorem 3.3 to the Novikov Conjecture is based on the following corollary. To make the connection visual, one should compare the corollary with the Ferry-Weinberger Descent Principle formulated in §5. Here (and there) H_*^{st} denotes the Steenrod homology.

Corollary 3.4. *Let X be a uniformly contractible n-manifold with a finite asymptotic dimension. Then there exists a Higson dominated metrizable corona N of X such that the boundary homomorphism $H_n^{lf}(X; \mathbf{Z}) = \mathbf{Z} \to H_{n-1}^{st}(N; \mathbf{Z})$ is a monomorphism.*

Proof. First we apply Theorem 3.3. Then the Schepin Spectral theorem [Dr1] implies that there is a metrizable Higson dominated corona N of X such that $\check{H}^*(X \cup N; \mathbf{Z}_p) = 0$ for all primes p. Then for the Steenrod homology the inclusion of the boundary induces an isomorphism $\tilde{\partial}_p : H_n^{lf}(X; \mathbf{Z}_p) \to H_{n-1}^{st}(N; \mathbf{Z}_p)$ for all p. Consider an $(n-2)$-connected

and locally $(n-2)$-connected compactum $Y = N \cup W$ with $\dim W \leq n-1$. Consider a diagram generated by exact sequence of pairs $(X \cup N, N) \subset (X \cup Y, Y)$. If the image in $H_{n-1}^{st}(Y)$ of the generator $1 \in \mathbf{Z} = H_n^{lf}(X)$ is not divisible by p, then it is not divisible by p in $H_{n-1}^{st}(N)$. Since W is $(n-1)$-dimensional, the inclusion homomorphism $H_{n-1}^{st}(N; \mathbf{Z}_p) \to H_{n-1}^{st}(Y; \mathbf{Z}_p)$ is a monomorphism. Hence, the boundary homomorphism $H_n^{lf}(X; \mathbf{Z}_p) \to H_{n-1}^{st}(Y; \mathbf{Z}_p)$ is a monomorphism. Since Y is locally $(n-2)$-connected, the Steenrod homology in the dimension $n-1$ agrees with the singular homology and we can use the universal coefficient formula. Since Y is $(n-2)$-connected, the universal coefficient formula implies that $H_{n-1}^{st}(Y; \mathbf{Z}_p) = H_{n-1}^{st}(Y) \otimes \mathbf{Z}_p$. Therefore the homomorphism $H_n^{lf}(X) \otimes \mathbf{Z}_p \to H_{n-1}^{st}(Y) \otimes \mathbf{Z}_p$ is a monomorphism. Hence the image of the generator is not divisible by p in $H_{n-1}^{st}(Y)$ and as well in $H_{n-1}^{st}(N)$. Therefore $\mathbf{Z} = H_n^{lf}(X) \to H_{n-1}^{st}(N)$ is a monomorphism. $\square$

REMARK. In [Ke] Keesling established that 1-dimensional cohomology of the Higson compactification of $\mathbf{R}^n$ is nonzero. Also it was shown in [D-F] that $\check{H}^n(\nu \mathbf{R}^n; \mathbf{Z}) \neq 0$. Then Theorem 3.3 implies that these groups are p-divisible for all p.

§4 Hypereuclidean manifolds

Let Y be a metric space with a base point y_0. We define the suspension ΣY as the quotient metric space of the product $I \times Y$ with the l_1-metric. Here the quotient map $q : I \times Y \to \Sigma Y$ collapses the set $\{0,1\} \times Y \cup I \times \{y_0\}$ to a point. The quotient metric is the maximal metric with respect to which q is 1-Lipschitz. Then the n-th suspension $\Sigma^n Y$ of Y can be identified inductively with the following quotient map:

$$q_n : I^n \times Y = I \times I^{n-1} \times Y \to I \times \Sigma^{n-1} Y = \Sigma^n Y.$$

Let $f : X \to Y$ be a map. We denote by $1_{I^n} \hat{\times} f = q_n \circ (1_{I^n} \times f) : I^n \times X \to \Sigma^n Y$. Note that $L(1_{I^n} \hat{\times} f) = L(f)$.

Lemma 4.1 (Trading Lemma). *For every n there is a number c_n such that for every L-Lipschitz map $f : I^n \to Y$ which is λ-Lipschitz on the boundary ∂I^n, the map $1_{I^n} \hat{\times} f : I^n \times I^n \to \Sigma^n Y$ can be deformed by a $c_n L$-Lipschitz homotopy $H : I^n \times I^n \times I \to \Sigma^n Y$, fixed on the boundary $\partial(I^n \times I^n)$, to a map g with $L(g|_{x \times I^n}) \leq c_n \lambda$ for all $x \in B^n$.*

Proof. We fix a c_n'-Lipschitz isotopy $H' : I^n \times I^n \times I \to I^n \times I^n$ that exchanges the factors. There is a $\bar{c}_n$-Lipschitz homeomorphism for some constant $\bar{c}_n$, which depends on n only,

$$\xi : (I^n \times I^n) \setminus Int \frac{1}{2}(I^n \times I^n) \to \partial(I^n \times I^n) \times I$$

such that the restriction of ξ to the boundary $\partial(I^n \times I^n)$ identifies it with $\partial(I^n \times I^n) \times \{0\}$ and the restriction of ξ to $\partial \frac{1}{2}(I^n \times I^n)$ is the multiplication by 2.

Let $c_n = 2c'_n \bar{c}_n$. We define

$$H(x,y,t) = \begin{cases} (1 \hat{\times} f) H'((1+t)x, (1+t)y, t) & if \ \ (x,y) \in \frac{1}{2}(I^n \times I^n) \\ (1 \hat{\times} f)(tH'(\xi(x,y)) + (1-t)(x,y)) & otherwise. \end{cases}$$

Note that

$$L((1 \hat{\times} f)H'((1+t)x, (1+t)y, t)) \leq 2c'_n L \leq c_n L$$

and

$$L((1 \hat{\times} f)(tH'(\xi(x,y)) + (1-t)(x,y))) \leq L(c'_n \bar{c}_n + 1) \leq c_n L.$$

Thus, the map H is $c_n L$-Lipschitz.

It is easy to check that $H(x,y,0) = 1 \hat{\times} f(x,y)$ for all $x,y \in I^n$.

If $(x,y) \in \partial(I^n \times I^n)$ we have that

$$H(x,y,t) = (1 \hat{\times} f)(tH'(\xi(x,y)) + (1-t)(x,y)) = (1 \hat{\times} f)(t(x,y) - (1-t)(x,y)) = (1 \hat{\times} f)(x,y).$$

Thus, the homotopy h_t is fixed on $\partial(I^n \times I^n)$.

Consider the map

$$g(x,y) = H(x,y,1) = \begin{cases} (1 \hat{\times} f)(2y, 2x) & if \ \ (x,y) \in \frac{1}{2}(I^n \times I^n) \\ (1 \hat{\times} f)H'(\xi(x,y)) & otherwise. \end{cases}$$

The Lipschitz constant of $(1 \hat{\times} f)(2y, 2x)$ as a function of y is equal 2. Since $L((1 \hat{\times} f)|_{\partial(I^n \times I^n)}) \leq \lambda$, we have that $L((1 \hat{\times} f) \circ H' \circ \xi) \leq c'_n \bar{c}_n \lambda \leq c_n \lambda$. Hence $L(g|_{x \times I^n}) \leq c_n \lambda$. $\quad \square$

For a map $f : X \times Z \to Y$ between metric spaces we denote

$$L^Z(f) = \sup\{L(f|_{X \times z}) \mid z \in Z\}, \quad L^X(f) = \sup\{L(f|_{x \times Z}) \mid x \in X\}.$$

The loop space ΩY on a metric space Y as any other mapping space is endowed with the natural sup metric. This makes the iterated loop space $\Omega^n Y$ equal to the mapping space $(Y, y_0)^{(I^n, \partial I^n)}$. Every map $f : I^n \times Z \to Y$ with $f(\partial I^n \times Z) = y_0$ induces a map $F : Z \to \Omega^n Y$ by the formula $F(z)(x) = f(x,z)$. If $L^{I^n}(f) \leq \lambda$, then $L(F) \leq \lambda$.

For every n and every Y there is a natural inclusion $j_n^Y : Y \to \Omega^n \Sigma^n Y$ defined by the following rule: A point $y \in Y$ corresponds under j_n^Y to the map $F_y = q_n \circ (1_{I^n} \times c_y) : I^n \to \Sigma^n Y$ where c_y is a constant map $c_y : I^n \to Y$ to y. One can check that j_n^Y is an isometric imbedding. Generally, one can define a map $j_{i,n+i}^Y : \Omega^i \Sigma^i Y \to \Omega^{n+i} \Sigma^{n+i} Y$ by taking a map $\phi : I^n \to \Sigma^n Y$ to the map $q_i(1_{I^i} \times \phi)$. Then $j_n^Y = j_{0,n}^Y$. Similarly $j_{i,n+i}^Y$ is an isometry. We note that $j_{i,n+k+i}^Y = j_{k+i,n+k+i}^Y \circ j_{i,k+i}^Y$ for all i, k, and n.

Corollary 4.2. *For every L-Lipschitz map $f : I^n \to Y$ with a λ-Lipschitz restriction on the boundary ∂I^n, the composition $j_n^Y \circ f : I^n \to \Omega^n \Sigma^n Y$ is homotopic to a $c_n \lambda$-Lipschitz map by means of $c_n L$-Lipschitz homotopy H_t which is fixed on ∂I^n.*

Proof. Consider a homotopy $h_t : I^n \times I^n \to \Sigma^n Y$ from Lemma 4.1. It defines a $c_n L$-Lipschitz homotopy $H_t : I^n \to \Omega^n \Sigma^n I^n$ by the formula $H_t(x)(z) = h_t(x, z)$. Then

$$H_0(x)(z) = h_0(x, z) = (i \hat{\times} f)(x, z) = q_n(x, f(z)) = F_{f(z)}(x) = j_n^Y(f(z))$$

Since $H_1(x)(z) = g(x, z)$ and $L^{I^n}(g) \le c_n \lambda$, we have $L(H_1(x)) \le c_n \lambda$. Since the homotopy h_t keeps the set $\partial(I^n \times I^n)$ fixed, the homotopy H_t keeps the boundary ∂I^n fixed. $\square$

By $\Omega_L^n W$ we denote the subset of $\Omega^n W$ consisting of L-Lipschitz maps $\phi : I^n \to W$.

For any numbers n and i denote by $\alpha_{n,i}^Y$ a map from $\Omega^n \Sigma^n \Omega^i \Sigma^i Y \to \Omega^{n+i} \Sigma^{n+i} Y$ defined by the following formula: $\alpha_{n,i}^Y(\Phi)(z, x) = q_{n+i}((y(z), x), \phi_z(x))$ where $\Phi(z) = q_n(y(z), \phi_z)$, $z \in I^n$, $x \in I^i$, $y : I^n \to I^n$, and $\phi_z : I^i \to \Sigma^i Y$.

Proposition 4.3. *The map $\alpha_{n,i}^Y$ is 1-Lipschitz.*

Proof. By the definition we have
$dist(\alpha_{n,i}^Y(\Phi), \alpha_{n,i}^Y(\Phi')) = \sup_{x,z} dist(\alpha_{n,i}^Y(\Phi)(z, x), \alpha_{n,i}^Y(\Phi')(z, x)) \le$
$\sup_{x,z} dist((y(z), x), \phi_z(x)), (y'(z), x), \phi_z'(x)) = \sup_{x,z} \|y(z) - y'(z)\| + \sup_{x,z} \|\phi_z(x) - \phi_z'(x)\| \le dist(\Phi, \Phi')$. $\square$

Proposition 4.4. $j_{i,n+i}^Y = \alpha_{n,i}^Y \circ j_n^{\Omega^i \Sigma^i Y}$ *for all i, n and Y.*

Proof. Let $\phi \in \Omega^i \Sigma^i Y$. Then $j_{i,n+i}^Y(\phi) = q_i(1_{I^n} \times \phi)$. On the other hand $\alpha_{n,i}^Y \circ j_n^{\Omega^i \Sigma^i Y}(\phi) = \alpha_{n,i}^Y(q_n(1_{I^n}, c_\phi)) = q_{n+i}(1_{I^n} \times 1_{I^i}, \times \phi) = q_i(1_{I^n} \times \phi)$. $\square$

Lemma 4.5. *For every natural number n there exists a number b_n such that for any positive λ, any two tending to infinity functions $\psi : \mathbf{R}_+ \to \mathbf{R}_+$, $\xi : \mathbf{R}_+ \to \mathbf{R}_+$ and every continuous map $f : K \to \Omega^i \Sigma^i Y$ of an n-dimensional uniform polyhedron with $L_f(t) \le \psi(t)$, $L(f|_{K^{(n-1)}}) \le \lambda$, and $f(x) \in \Omega_{\xi(\|x\|)}^i \Sigma^i Y$ for all $x \in K$, there exists a homotopy $H : K \times I \to \Omega^{n+i} \Sigma^{n+i} Y$ deforming a map $j_{i,n+i}^Y \circ f$ to a map g with the properties:*

  (1) $L(g) \le b_n \lambda$;
  (2) $L_H(t) \le b_n \psi(t)$ *and*
  (3) $g(x) \in \Omega_{2b_n(\psi(\|x\|) + \xi(\|x\|))}^{n+i} \Sigma^{n+i} Y$.

Proof. We fix an n-simplex $\Delta \subset K$ and consider the restriction f_Δ of f to it. Note that $L(f_\Delta|_{\partial\Delta}) \leq \lambda$ and $L(f_\Delta) \leq \psi(\|\Delta\|)$, where $\|\Delta\| = \max\{dist(x, x_0) \mid x \in \Delta\}$. Since Δ is a_n-Lipschitz homeomorphic to the cube I^n, we may apply Corollary 4.2. According to Corollary 4.2 there is a $a_n c_n L(f_\Delta)$-Lipschitz homotopy in $\Omega^n \Sigma^n (\Omega^i \Sigma^i Y)$ from $j_n^{\Omega^i \Sigma^i Y} \circ f$ to a $a_n c_n \lambda$-Lipschitz map g_Δ which is fixed on the boundary $\partial\Delta$. The union of these homotopies defines a homotopy $\tilde{H} : K \times I \to \Omega^n \Sigma^n (\Omega^i \Sigma^i Y)$ from $j_n^{\Omega^i \Sigma^i Y} \circ f$ to a $a_n c_n \lambda$-Lipschitz map $\tilde{g}$ satisfying the inequality $L_{\tilde{H}}(t) \leq a_n c_n \psi(t)$. We define $H = \alpha_{n,i}^Y \circ \tilde{H}$. Then H is a homotopy between $g = \alpha_{n,i}^Y \circ \tilde{g}$ and $\alpha_{n,i}^Y \circ j_n^{\Omega^i \Sigma^i Y} \circ f$. The latter is equal to $j_{i,n+i}^Y \circ f$ by Proposition 4.4. The condition (1) holds for $\tilde{g}$ with $b_n = a_n c_n$ by the construction. Then by Proposition 4.3 it holds for g. Similarly, the condition (2) holds for $\tilde{H}$ and by Proposition 4.3 it holds for H.

Let $\tilde{g}(x) = \Phi \in \Omega^n \Sigma^n \Omega^i \Sigma^i Y$. Note that $\Phi(z) = q_n(y(z), \phi_z)$, where $y : I^n \to I^n$ and $\phi_z : I^i \to \Sigma^i Y$ are two maps. By the construction of $\tilde{g}$ we have $\tilde{g}(x) \in \Omega_{b_n \psi(\|x\|)}^n \Sigma^n \Omega_{\xi(\|x\|)}^i \Sigma^i Y$. It means that the maps y and ϕ_z are $b_n \psi(\|x\|)$-Lipschitz as functions of z. Additionally, the map ϕ_z is $\xi(\|x\|)$-Lipschitz for every z. Let $(z, t), (z', t') \in I^n \times I^i = I^{n+i}$. Then

$$dist(\alpha_{n,i}^Y(\Phi)(z, t), \alpha_{n,i}^Y(\Phi)(z', t')) \leq dist(q_{n+i}(y(z), t), \phi_z(t)), q_{n+i}(y(z'), t'), \phi_{z'}(t'))$$

$$\leq dist((y(z), t), \phi_z(t)), (y(z'), t'), \phi_{z'}(t')) = \|y(z) - y(z')\| + \|t - t'\| + dist(\phi_z(t), \phi_{z'}(t'))$$

$$\leq b_n \psi(\|x\|)\|z - z'\| + \|t - t'\| + dist(\phi_z(t), \phi_z(t')) + dist(\phi_z(t'), \phi_{z'}(t')) \leq$$

$$\max\{b_n \psi(\|x\|), 1\}\|(z, t) - (z', t')\| + \xi(\|x\|)\|t - t'\| + b_n \psi(\|x\|)\|z - z'\| \leq$$

$2b_n(\psi(\|x\|) + \xi(\|x\|))$. Here we assume that $\psi \geq 1$. Thus, the condition (3) is checked. $\square$

Let $f : K \to Y$ be a map of an infinite simplicial complex to a metric space which is Lipschitz on every finite complex and let $x_0 \in K$ be a base point. We define $L_f((t) = L(f|_{B_t(x_0)})$, where $B_t(x_0)$ is a ball of radius t centered at x_0. For a homotopy $H : K \times I \to Y$ we denote by $L_H(t) = L(f|_{B_t(x_0) \times I})$.

Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a function, by $B_\phi^n \times K$ we denote the set $\{(z, x) \in \mathbf{R}^n \times K \mid \|z\| \leq \phi(d_K(x, x_0))\}$.

Lemma 4.6. *Let K be a finite dimensional uniform polyhedron and let $f : K \to Y$ be a continuous map to a compact polyhedron Y with $L_f(t) \leq \psi(t)$ for some function ψ. Then there are numbers n, a and λ and a map $g : K \to \Omega^n \Sigma^n Y$ with $g(x) \in \Omega_{a\psi(\|x\|)}^n \Sigma^n Y$*

satisfying the inequalities $L(g) < \lambda$ which is homotopic to the map $j_n^Y \circ f$ by means of a homotopy H with the property $L_H(t) \leq a\psi(t)$.

Proof. We apply induction on $\dim K = i$. If $\dim K = 0$, then we assume that all points in K are at least one apart. Then the statement follows from compactness of Y with $n = 0$, $a = 1$ and $\lambda = 2diam(Y)$.

Assume that the lemma holds for i and let $\dim K = i+1$. Let $f : K \to Y$ be given with $L_f(t) < \psi(t)$. By the inductive assumptions, there is a map $g_i : K^{(i)} \to \Omega^r \Sigma^r Y$ with $L(g_i) < \lambda_i$, $g_i(x) \in \Omega^r_{a_i\psi(\|x\|)}\Sigma^r Y$, and a homotopy H^i, joining $j_r^Y \circ f$ with g_i, which satisfies the inequality $L_{H^i}(t) \leq a_i\psi(t)$. We consider an arbitrary $(i+1)$-simplex $\Delta \subset K$ and consider the map f restricted to Δ. Then the homotopy H^i on $\partial\Delta$ can be extended to a homotopy $\bar{H}^i$ of Δ by means of an isotopy $G : \Delta \times I \to \Delta \times \{0\} \cup \partial\Delta \times I$, which joins the embedding Δ as $\Delta \times \{0\}$ to a homemomorphism between Δ and $\Delta \times \{0\} \cup \partial\Delta \times I$. We may assume that G is c-Lipschitz where c depends on i only. Let $f_1 = \bar{H}_1^i : \Delta \to \Omega^r \Sigma^r Y$. Note that $L(\bar{H}^i) < cL_f(d(\Delta, x_0))$ and $L(f_1|_{\partial\Delta}) = L(g_i) < \lambda_i$. Let $\tilde{H}$ be the union of homotopies $\bar{H}^i$ for all $i+1$-simplices Δ and let $\tilde{f} = \tilde{H}_1$. We note that $L_{\tilde{H}}(t) \leq cL_f(t) \leq c\psi(t)$ and $L(\tilde{f}|_{K^{(i)}}) < \lambda_i$. Also we note that $\tilde{f}(x) \in \Omega^r_{a_i\psi(\|x\|)}\Sigma^r Y$. We apply Lemma 4.5 to the map $\tilde{f} : K \to \Omega^r \Sigma^r Y$ with $n = i+1$ to obtain a homotopy H' transforming $j_{r,i+1+r}^Y \tilde{f}$ to a map $g : K \to \Omega^{i+1+r}\Sigma^{i+1+r}Y$ with the properties: (1) $L(g) \leq b_n\lambda_i$; (2) $L_{H'}(t) \leq b_nc\psi(t)$; (3) $g(x) \in \Omega^{i+1+r}_{2b_n(1+a_i)\psi(\|x\|)}\Sigma^{i+1+r}Y$. We set $\lambda = b_n\lambda_i$ and $a = 2b_n \max\{2(1+a_i), c\}$ and define a homotopy H as $j_{r,i+1+r}^Y \tilde{H}$ followed by H'. Note that H joins $j_{r,i+1+r}^Y \circ j_r^Y \circ f = j_{i+1+r}^Y$ with g. Then all conditions are satisfied. $\square$

Corollary 4.7. *Let K and $f : K \to Y$ be as above with $L_f(t)\psi(t)$ where ψ is a Lipschitz function. Then there are a number n and a Lipschitz map $q : B_\psi^n \times K \to \Sigma^n Y$ such that q is homotopic to $(1\hat{\times}f)\circ(\frac{1}{\psi(\| \|)} \times 1_K)$ by means of a homotopy h_t with $h_t(\partial B_\psi^n \times K) = y_0$.*

Proof. Let $g : K \to \Omega^n\Sigma^n Y$ be given by Lemma 4.6. The map g defines a map $\hat{g} : B^n \times K \to \Sigma^n Y$ with the properties: (1) $L^{B^n}(\hat{g}) \leq \lambda$; (2) $L(\hat{g}|_{B^n \times \{x\}}) \leq a\psi(\|x\|)$; (3) $\hat{g}$ is homotopic to $1\hat{\times}f$ by means of homotopy h_t' with $h_t'(\partial B_{\psi(t)}^n \times K) = y_0$.

We define $q(b, x) = \hat{g}(\frac{b}{\psi(\|x\|)}, x)$, where $x \in K$, $b \in B_{\psi(\|x\|)}^n$. We show that q is Lipschitz. First we consider a pair of points $(b_1, x), (b_2, x) \in B_\psi^n \times K \subset \mathbf{R}^n \times K$. Note that

$$d(q(b_1, x), q(b_2, x)) \leq d(\hat{g}(b_1/\psi(\|x\|), x), \hat{g}(b_2/\psi(\|x\|), x)) \leq L_g(\|x\|)\|b_1/\psi(\|x\|) - b_2/\psi(\|x\|)\|.$$

Then by the conclusion of Lemma 4.6 we have

$$d(q(b_1, x), q(b_2, x)) \le a\psi(\|x\|)\|\frac{b_1}{\psi(\|x\|)} - \frac{b_2}{\psi(\|x\|)}\| \le a\|b_1 - b_2\|.$$

Next, we consider a pair of type (b, x_1) and (b, x_2). Then

$$d(q(b, x_1), q(b, x_2)) = d(\hat{g}(\frac{b}{\psi(\|x_1\|)}, x_1), \hat{g}(\frac{b}{\psi(\|x_2\|)}, x_2)) \le$$

$$d(\hat{g}(\frac{b}{\psi(\|x_1\|)}, x_1), \hat{g}(\frac{b}{\psi(\|x_1\|)}, x_2)) + d(\hat{g}(\frac{b}{\psi(\|x_1\|)}, x_2), \hat{g}(\frac{b}{\psi(\|x_2\|)}, x_2))$$

$$\le \lambda d(x_1, x_2) + a\psi(\|x_2\|)\|b/\psi(\|x_1\|) - b/\psi(\|x_2\|)\|$$

$$\le \lambda d(x_1, x_2) + a|\psi(\|x_2\|) - \psi(\|x_1\|)\|\| \le (\lambda + a\mu)d(y_1, y_2)$$

where μ is a Lipschitz constant for ψ. $\square$

Let (K, d) be an asymptotic polyhedron and let $u_K : K \to K_U$ denote the identity map which switch from the metric d to the uniform metric. Let $f : \mathbf{R}_+ \to \mathbf{R}_+$ be a tending to infinity function, we say that K has growth less than f if $\|x\|_d < f(\|x\|_U)$. We say that K has a *quadratic growth* if $\|x\|_d < \|x\|_U^2$.

Lemma 4.8. *Let X be the universal cover of an aspherical manifold M_0 , $p : X \to M_0$, supplied with the induced metric, and let $\operatorname{asdim} X = n$. Then, given a tending to infinity function $\beta : \mathbf{R}_+ \to \mathbf{R}_+$, there are an n-dimensional asymptotic polyhedron N, a proper 1-Lipschitz map $\phi : X \to N$, and a proper homotopy inverse map $\gamma : N \to X$ with $L_{\gamma \circ u_K^{-1}}(t) < \beta(t)$, where $K \subset N$ and the closure of $N \setminus K$ is compact.*

Proof. First we show that for the projection $\phi_{\mathcal{U}} : X \to N(\mathcal{U})$ to the uniform nerve of a uniformly bounded cover $\mathcal{U}$ admits a Lipschitz homotopy inverse $\gamma : N(\mathcal{U}) \to X$. We define γ on the vertices v by taking points $\gamma(v) \in \phi_{\mathcal{U}}^{-1}(v)$ and define γ on every edge $[v, u]$ by sending it with a constant speed along a minimal geodesic joining $\gamma(v)$ and $\gamma(u)$. Assume that a λ_i-Lipschitz map $\gamma : N^{(i)}(\mathcal{U}) \to X$ is already constructed for $i \ge 1$. For every $(i + 1)$-dimensional simplex Δ we consider the map $f = p \circ \gamma|_\Delta$. Since this map is null homotopic and $\pi_{i+1}(M_0) = 0$, Lemma 2.3 implies that there exists λ_{i+1}-Lipschitz extension ξ_Δ, $\lambda_{i+1} = \mu(\lambda_i)$. Then there is a unique lift ξ' of ξ that extends $\gamma|_{\partial\Delta}$. Since p is a local isometry, the lift ξ' is also λ_{i+1}-Lipschitz. The union of all ξ' defines a λ_{i+1}-Lipschitz map on the $(i + 1)$-skeleton.

For a homotopy lift $\gamma : N \to X$ from Lemma 3.2 we define a function $\psi_\gamma(t) = L(\gamma|_{B_t(x_0)})$. Because of the above remark we always may assume that the function ψ_γ takes finite values. Moreover, for any given tending to infinity function $\beta : X \to \mathbf{R}_+$

there are an approximation $\phi : X \to N$ by an asymptotic n-dimensional polyhedron and a proper homotopy inverse map $\gamma : N \to X$ with $\psi_\gamma < \beta$.

For every approximation by an asymptotic polyhedron one can find better approximation by an asymptotic polyhedron of the quadratic growth. To do that we enlarge the sets C_i in the proof of Lemma 3.1 so that C_i contains the set $\phi_i^{-1}(B_{2^i}(x_0))$, where $\phi_i : X \to N(\mathcal{U}_i)$ is the projection to the nerve with the uniform metric. Then for every $x \in B_{2^{i+1}}(x_0) \setminus B_{2^i}(x_0)$ we have $\|x\|_U = \Sigma_{k=1}^i 2^i + c$ where c is the length of a segment in $B_{2^{i+1}}(x_0) \setminus B_{2^i}(x_0)$. We note that

$$\|x\|_d \leq \Sigma_{k=1}^i 2^i 2^i + 2^{i+1}c \leq (\Sigma_{k=1}^i 2^i + c)^2 = \|x\|_U^2.$$

In view of this we can find an arbitrarily close approximation $\phi : X \to N$ by asymptotic polyhedra of the quadratic growth. According to the above remark we can take an approximation such that there exists a homotopy inverse map $\gamma : N \to X$ with $L_\gamma(t) < \beta(\sqrt{t})$. Then

$$L_{\gamma \circ u_K^{-1}}(t) = L(\gamma \circ u_K^{-1}|_{B_t^U(x_0)}) \leq L(\gamma|_{u_K^{-1}(B_t^U(x_0))}) \leq L(\gamma|_{B_{t^2}(x_0)}) = L(\gamma)(t^2) < \beta(t)$$

$\square$

DEFINITION. An m-manifold M is called integrally *hypereuclidean* if there is a proper Lipschitz map $f : M \to \mathbf{R}^m$ of degree one.

Theorem 4.9. *Let M be the universal cover of a closed aspherical m-manifold with* asdim $M < \infty$. *Then the product $M \times R^n$ is integrally hypereuclidean for some k.*

Proof. There is a Lipschitz map $\alpha : (M \times \mathbf{R}) \setminus C \to S^m$ of degree one for some compact set C. Indeed, we may assume that for some ball B_r in M there is a Lipschitz map $\gamma : (B_r, \partial B_r) \to (B_r/\partial B_r, \{\partial B_r\}) \to (S^m, y_0)$ to the unit sphere which is a relative homeomorphism. Let $C = B \times [-1, 1]$. One can take

$$\alpha(x, t) = \begin{cases} \gamma(x) & if \;\; x \in B_r \;\; and \;\; t \geq 1 \\ y_0 & otherwise \end{cases}$$

Let $L(\alpha) < a$. Let $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ be a Lipschitz map of a sublinear growth.

Let $\phi : M \times \mathbf{R} \to N$ be an approximation by an asymptotic polyhedron of quadratic growth as in Lemma 4.8 with a homotopy inverse map $\gamma : N \to M \times \mathbf{R}$ with $L_\gamma(t) \leq \max\{\psi(\sqrt{t})/a, c\}$. We can find a subpolyhedron $K \subset L$ with a relatively compact complement such that $L_{\gamma|_K}(t) \leq \psi(\sqrt{t})/a$ and the map $\alpha \circ \gamma|_K \circ \phi|_{\gamma^{-1}(K)}$ induces the same element $[\alpha|_{\gamma^{-1}(K)}]$ in the m-dimensional cohomology. Let $f = \alpha \circ \gamma|_K \to S^m$,

where K is considered as a uniform complex. Then $L_f(t) \le \psi(t)$. Then by the Corollary 4.7 there is n and a Lipschitz map $q : B^n_{\psi(t)} \times K \to S^{n+m}$ such that q is homotopic to $(1 \hat{\times} f) \circ (\frac{1}{\|\ \|} \times 1_K)$ by means of a homotopy h_t with $h_t(\partial B^n_t \times K) = y_0$.

We define a map $\xi : \mathbf{R}^n \to \mathbf{R}^n$ as $\xi(z) = z/\psi(\|z\|)$ and consider the map $g = q \circ (\xi|_{B^n_t} \times u_K) : B^n_t \times K \to S^{n+m}$ where K is taken with its metric of an asymptotic polyhedron. It is easy to see that the map g has the Lipschitz constant vanishing at infinity. We extend the map g to a map $\bar{g} : \mathbf{R}^n \times K \to S^{n+m}$ by the constant map. Then the map

$$w = \bar{g} \circ (1_{\mathbf{R}^n} \times \phi|) : \mathbf{R}^n \times ((M \times \mathbf{R}) \setminus C) \to S^{n+m}$$

can be continuously extended to $\bar{w}$ by the constant map over $(\mathbf{R}^n \setminus B^n_R) \times C$ for a large enough R. Then $\bar{w}$ has the gradient tending to zero at infinity. Therefore w can be extended over the Higson corona $\nu(\mathbf{R}^{n+1} \times M)$. We note that the map $\bar{w}$ has degree one. Then it follows that $\mathbf{R}^k \times M$ is integrally hypereuclidean for $k = n + 1$ [Ro1]. $\quad\square$

§5 Application to the Novikov Conjecture

There are several different coarse type conditions on a manifold M that apply the Novikov Higher Signature Conjecture. A comparison of some of them was attempted in [Dr1]. We will use here the following refinement of Carlson-Pedersen's conditions [C-P], [F-W] which is due to Ferry-Weinberger:

Descent Principle [D-F]. *Let $p : X \to M$ be the universal cover of a closed aspherical n-manifold with the fundamental group $\Gamma = \pi_1(M)$. Suppose that X admits a Γ-equivariant metrizable compactification $\bar{X} = X \cup \partial X$ such that the boundary homomorphism $H^{lf}_n(X; \mathbf{Q}) \to H^{st}_{n-1}(\partial X; \mathbf{Q})$ is a Γ-equivariant split monomorphism for the Steenrod homology. Then the Novikov conjecture holds for M.*

As we mentioned in Introduction, G. Yu proved the coarse Baum-Connes conjecture for all geometrically finite groups with finite asymptotic dimension [Yu1]. In particular his result implies the Novikov conjecture for these groups [Ro2]. Here we give an alternative prove of the Novikov conjecture based on the Ferry-Weinberger Descent Principle.

Theorem 5.1. *Suppose that the fundamental group $\Gamma = \pi_1(M)$ of a closed aspherical manifold M has a finite asymptotic dimension as a metric space with the word metric. Then the Novikov conjecture holds for M.*

We note that an action of a group Γ by isometries on a metric space X can be extended over the Higson corona.

DEFINITION. An n-manifold X with an isometric action of a group Γ on it is called Γ-*hypereuclidean* if its Higson corona νX admits a map $f : \nu X \to S^{n-1}$ of degree one

which induces a Γ-equivariant homomorphism of the cohomologies for the trivial action on S^{n-1}. In other words f defines a cohomology element fixed under the action of Γ on $\check{H}^{(n-1)}(\nu X)$. By the definition the degree of f is the degree of the homomorphism

$$\mathbf{Z} \to H^{n-1}(S^{n-1}) \xrightarrow{f^*} \check{H}^{n-1}(\nu X) \xrightarrow{\delta} H_c^n(X) = \mathbf{Z}.$$

The proof of Theorem 5.1 is based on the following two lemmas.

Lemma 5.2. *Suppose that the universal cover X of a closed aspherical m-manifold M is equivariantly hypereuclidean. Then the Novikov conjecture holds for M.*

Proof. Let $f : X \to \mathbf{R}^m$ be a proper Lipschitz map of degree one such that the extension map $\bar{f}$ to the Higson corona is a cohomology equivariant map to S^{m-1}. Consider the Γ-equivariant commutative diagram induced by this extension and the boundary maps

$$
\begin{array}{ccc}
\check{H}^{m-1}(\nu X; \mathbf{Q}) & \xrightarrow{\;\;\delta\;\;} & H_c^m(X; \mathbf{Q}) \\[4pt]
\bar{f}^* \uparrow & & f^* \uparrow \\[4pt]
H^{m-1}(S^{m-1}; \mathbf{Q}) & \xrightarrow{\;\;\tilde{\delta}\;\;} & H_c^m(\mathbf{R}^m; \mathbf{Q})
\end{array}
$$

Here the homomorphism $\delta \circ \bar{f}^*$ is an equivariant isomorphism. Therefore the homomorphism δ is a Γ-equivariant split epimorphism.

Then by the Shchepin spectral theorem one can find a Γ-equivariant metrizable compactification $\bar{Y} = Y \cup \partial Y$ such that the boundary homomorphism $\bar{\delta} : \check{H}^{m-1}(\partial Y; \mathbf{Q}) \to H_c^m(Y; \mathbf{Q})$ is a Γ-equivariant split monomorphism (see the proof of Lemma 9.3 in [Dr1]). Then the boundary homomorphism for the Steenrod homology

$$\bar{\partial} : H_m^{lf}(Y; \mathbf{Q}) \to H_{m-1}^{st}(\partial Y; \mathbf{Q})$$

is a Γ-equivariant split monomorphism. The Descent Principle completes the proof. $\square$

Lemma 5.3. *Let $\Gamma = \pi_1(M)$ be the fundamental group of an aspherical m-manifold and let X be its universal cover. Assume that $\mathrm{asdim}\, X < \infty$. Then there is a number N such that $X \times \mathbf{R}^N$ is Γ-hypereuclidean*

Proof. We fix an isometry $h : X \to X$ that preserves the orientation. We consider a Lipschitz map $\alpha : X \times \mathbf{R} \setminus C \to S^m$ from the proof of Theorem 4.9. First, we note that $\alpha|_{X \times \mathbf{R} \setminus C'}$ is Lipschitz homotopic to $\alpha \circ (h \times 1)|_{X \times \mathbf{R} \setminus C'}$, where C' is a compact set in $X \times \mathbf{R}$ that contains C and $h^{-1}(C)$. The proof of Theorem 4.9 produces a map of the degree one $w : \mathbf{R}^n \times (X \times \mathbf{R} \setminus C') \to S^{n+m}$ extendable over the Higson corona,

and a Lipschitz homotopy H, connecting $1\hat{\times}\alpha$ with w. Then $H \circ (1_{\mathbf{R}^n} \times h \times 1_{\mathbf{R}})$ is a Lipschitz homotopy connecting $1\hat{\times}\alpha \circ (h \times 1)$ with $w \circ (1_{\mathbf{R}^n} \times h \times 1_{\mathbf{R}})$. It means that the maps w and $w \circ (1_{\mathbf{R}^n} \times h \times 1_{\mathbf{R}})$ are Lipschitz homotopic. Then, like in the proof of Theorem 4.9, we can show that there is a Lipschitz homotopy H' between $1_{\mathbf{R}^k} \hat{\times} w$ and $1_{\mathbf{R}^k} \hat{\times} w \circ (1_{\mathbf{R}^n} \times h \times 1_{\mathbf{R}})$ for some k which is extendable over the Higson corona for every $t \in [0,1]$. This homotopy defines a map

$$G : \mathbf{R}^{k+n} \times (X \times \mathbf{R} \setminus C') \to (S^{k+n+m})_\lambda^I$$

to the space of all λ-Lipschitz mappings of the interval I to the sphere S^{k+n+m} for some λ which is compact. Then it follows that G is extendible over the Higson corona. Therefore, G defines a homotopy $\bar{G} : \nu(X \times \mathbf{R}^{n+k+1}) \times I \to S^{k+n+m}$ between the extensions $\tilde{w}$ and $\tilde{w}_h$ of the maps $1\hat{\times}w$ and $1\hat{\times}(w \circ (1 \times h)$ over the Higson corona. This means that the map $\tilde{w}$ is cohomologically h-invariant.

Let $h_i : X \to X$, $i = 1,\ldots,l$, be isometries on X defined by generators of Γ. We apply the above argument to each h_i and take maximal k. Then $N = k + n + 1$. $\square$

Proof of Therem 5.1. Since the action of $\mathbf{Z}^N$ on $X \times \mathbf{R}^N$ induces the trivial action on $\nu(X \times \mathbf{R}^N)$, the product $X \times \mathbf{R}^N$ is Γ'-hypereuclidean for $\Gamma' = \Gamma \times \mathbf{Z}^N$. Then by Lemma 5.2 the Novikov Conjecture holds for Γ' and hence for Γ. $\square$

Corollary 5.4. *Suppose that the classifying space $B\Gamma$ is a finite complex for a group Γ with $\operatorname{asdim}\Gamma < \infty$. Then the Novikov conjecture holds for Γ.*

Proof. For every group Γ with finite $B\Gamma$ M. Davis [D] gave a construction of an aspherical closed manifold M such that $B\Gamma$ is a retract of M. In [B-D] (Theorem 8) we proved that $\operatorname{asdim} M = \operatorname{asdim} \pi_1(M) < \infty$ provided $\operatorname{asdim}\Gamma < \infty$.

It is well-known that the Novikov Conjecture for $\pi_1(K)$ is equivalent to the injectivity of the assembly map A_K [F-R-R]. Hence, by Theorem 5.1, A_M is a monomorphism. We consider the following diagram:

$$
\begin{array}{ccc}
H_*(B\pi_1(K);\mathbf{Q}) & \xrightarrow{A_K} & L_*(\mathbf{Z}[\pi_1(K)]) \otimes \mathbf{Q} \\
i_* \downarrow & & \downarrow \\
H_*(B\pi_1(M);\mathbf{Q}) & \xrightarrow{A_M} & L_*(\mathbf{Z}[\pi_1(M)]) \otimes \mathbf{Q}
\end{array}
$$

Since the inclusion $i : K \to M$ admits a retraction, it follows that i^* is a monomorphism. Since A_K is a left divisor of a monomorphism $A_M \circ i_*$, it is a monomorphism itself. $\square$

In the conclusion of this section we formulate a problem which can be considered as a stable version of the Higson conjecture [Ro1]. We recall that the original unstable version is incorrect even in the cases where the Novikov conjecture is true [Ke].

Let

$$p : \overline{X \times \mathbf{R}} = X \times \mathbf{R} \cup \nu(X \times \mathbf{R}) \to X \times \mathbf{R} \cup \Sigma \nu X = \Sigma \bar{X}$$

be the natural projection of compactifications of $X \times \mathbf{R}$. We define a group

$$SH^i(X) = \varinjlim \{ \check{H}^i(\bar{X}) \xrightarrow{\Sigma} \check{H}^{i+1}(\Sigma \bar{X}) \xrightarrow{p^*} \check{H}^i(\overline{X \times \mathbf{R}}) \to \dots \}.$$

Problem 5.5. *Is $SH^*(X) = 0$ for the universal cover X of a closed aspherical manifold M?*

We note that the affirmative answer to this problem implies the Novikov conjecture for M.

§6 Asymptotically piecewise Euclidean spaces and the Gromov-Lawson conjecture

We recall that a metric space X has a *bounded geometry* if for every $\epsilon > 0$ and $R > 0$ there is a number C such that for every $x \in X$ every ϵ-net in the ball $B_R(x)$ contains no more than C points. A finitely generated group with the word metric or the universal cover of a finite complex with the lifted metric are typical examples of metric spaces of bounded geometry. Every metric space X of bounded geometry for arbitrarily large λ admits a uniformly bounded cover of a finite multiplicity with the Lebesgue number $\geq \lambda$. Like in the proof of Lemma 3.1 we can conclude that every metric space of bounded geometry has the following property:

(**) *For arbitrarily large λ there is $n = n(\lambda)$, n-dimensional uniform simplicial complex K_λ of size λ, supplied with l_∞-metric, and a 1-Lipschitz uniformly cobounded map $\phi_\lambda : X \to K_\lambda$.*

A map of a metric space to a simplicial complex $f : X \to K$ is called uniformly cobounded if there is a constant C such that $\operatorname{diam} \phi^{-1}(\Delta) \leq C$ for all simplices Δ. The property (**) can be organized in so called an anti-Čech approximation of X [H-R1]. An *anti-Čech approximation* of X is a direct system of finite dimensional uniform simplicial complexes with 1-Lipschitz bonding maps $\{K_{\lambda_n}, g_n^m\}$ with $\lambda_n \to \infty$ together with a sequence of 1-Lipschitz uniformly cobounded maps $\phi_{\lambda_n} : X \to K_{\lambda_n}$ such that $\phi_{\lambda_n} = g_n^m \phi_{\lambda_m}$ for all $n > m$. We call an anti-Čech approximation *Euclidean* if all complexes K_{λ_n} are given the Euclidean metric.

We say that X admits a Čech approximation by a certain class $\mathcal{C}$ of asymptotic polyhedra if given a proper function $f : X \to \mathbf{R}_+$ there is an asymptotic polyhedron

$N \in \mathcal{C}$ and an 1-Lipschitz map $\phi : X \to N$ with $\operatorname{diam} \phi^{-1}(\Delta) \leq \min_{x \in \Delta} f(x)$ for every simplex $\Delta \subset N \setminus C$ for some compact set C. We recall that Lemma 3.1 asserts that if $\operatorname{asdim} X < \infty$, then X can be approximated by n-dimensional Euclidean asymptotic polyhedra K with the following properties:

1) Every simplex in K is either the standard Δ_{2^i} of size 2^i for some i or it isomorphic to the simplex $\tilde{\Delta}_{2^i}$ with edges of two types: of length 2^i and 2^{i+1};

2) If two simplices in K have nonempty intersection then they either of the same type or they have the types Δ_{2^i} and $\tilde{\Delta}_{2^i}$ or they have the types Δ_{2^i} and $\tilde{\Delta}_{2^{i-1}}$.

The class of Euclidean asymptotic polyhedra with the above property we denote by $\mathcal{C}_0$. The arguments of Lemma 3.1 and Lemma 3.2, applied in the infinite dimensional case, gives us the following

Proposition 6.1. *If a metric space of bounded geometry X admits an Euclidean anti-Čech approximation, then it can be approximated by Euclidean asymptotic polyhedra from the class $\mathcal{C}_0$.*

Proposition 6.2. *If a metric space of bounded geometry X admits an Euclidean anti-Čech approximation, then it admits an approximation by Euclidean asymptotic polyhedra $\phi : X \to K$ from the class $\mathcal{C}_0$ with proper homotopy inverse maps $\gamma : X \to K$.*

DEFINITION. A metric space that admits a Čech) approximation by Euclidean asymptotic polyhedra is called *asymptotically piecewise Euclidean.*

We note that if X admits an approximation by Euclidean asymptotic polyhedra then it admits an approximation by asymptotic polyhedra from the class $\mathcal{C}_0$.

We recall that a *uniform embedding* in the coarse sense is a map $q : X \to Y$ between metric spaces such that

$$\rho_1(d_X(x, y)) \leq d_Y(q(x), q(y)) \leq \rho_2(d_X(x, y))$$

for two tending to infinity functions $\rho_1, \rho_2 : \mathbf{R}_+ \to \mathbf{R}_+$ and all $x, y \in X$.

Lemma 6.3. *Every Euclidean asymptotic polyhedron $K \in \mathcal{C}_0$ admits a uniform embedding in the Hilbert space l_2.*

Proof. The uniform simplicial complex K_U can be naturally embedded into the infinite simplex $\Delta^\infty = \{(t_1, t_2, \dots) \in l_2 \mid \Sigma t_i = 1, t_i \geq 0\}$. Let Δ be a simplex in K and let $l(\Delta)$ denote the minimum of the length of edges in Δ. We define a map $l : K \to \mathbf{R}_+$ by the rule: $l(x) = \min\{l(\Delta) \mid x \in \Delta\}$. Let $\rho : K \to \mathbf{R}_+$ be an 1-Lipschitz function tending to infinity and $l/2 \leq \rho \leq l$. We define a map $q : K \to l_2$ by the formula $q(x) = \rho(x)u(x)$ and show that it is a uniform embedding.

First we show that q is 3-Lipschitz. Since a metric on K is geodesic, it suffices to show that q is 3-Lipschitz on every simplex. For every two points x and y from the same simplex Δ we have

$$\|q(x)-q(y)\| = \|\rho(x)u(x)-\rho(y)u(y)\| \le \|\rho(x)u(y)-\rho(y)u(y)\|+\|\rho(x)u(x)-\rho(x)u(y)\| \le$$

$$d_K(x,y)\|u(y)\| + \rho(x)\|u(x)-u(y)\| \le d_K(x,y) + \rho(x)d_U(x,y).$$

Since the projection of the simplices Δ_l and $\tilde{\Delta}_l$ onto the standard unit simplex is $(1/l)$-Lipschitz, we have

$$d_U(x,y) \le (2/l(\Delta))d_K(x,y) \le (2/l(x))d_K(x,y) \le (2/\rho(x))d_K(x,y).$$

Then $\|q(x) - q(y)\| \le 3d_K(x,y)$.

Now assume that q^{-1} is not uniform. It means that there is a number $c > 0$ and a sequence of pairs of points $\{(x_k, y_k)\}$ in K such that $d_K(x_k, y_k) \to \infty$ and $\|q(x_k) - q(y_k)\| \le c$. We may assume that $\rho(x_k) \le \rho(y_k)$ for all k . Then $c/\rho(x_k) \ge \|u(x_k) - u(y_k)\|$. Since $\rho(x_k) \to \infty$, for large enough k we have the inequality $\|u(x_k) - u(y_k)\| \ge (1/2)d_U(x_k, y_k)$. Moreover, we may assume that x_k and y_k lie in two simplices Δ_1, Δ_2 in K with $\Delta_1 \cap \Delta_2 \ne \emptyset$. Then $2\max\{l(\Delta_1), l(\Delta_2)\}d_U(x_k, y_k) \ge d_K(x_k, y_k)$. Therefore, $4\max\{l(x), l(y)\}d_U(x_k, y_k) \ge d_K(x_k, y_k)$. Then $16\rho(x)d_U(x_k, y_k) \ge d_K(x_k, y_k)$ and hence we get a contradiction: $32c \ge 16\rho(x)d_U(x_k, y_k) \ge d_K(x_k, y_k)$ for all k. $\square$

Clearly, every discrete subspace of the Hilbert space is asymptotically Euclidean.

Theorem 6.4. *For an asymptotically Euclidean geometrically finite groups Γ the Roe index map is a split monomorphism.*

Proof. Let $X = E\Gamma$ and let $\phi : X \to K$ and $\gamma : K \to X$ as in Proposition 6.2. According to Lemma 6.3 the asymptotic polyhedron K is uniformly embeddable into l_2. By the theorem of Yu [Yu2] the coarse Baum-Connes conjecture holds for K, i.e. the Roe index map $A_K : K_*^($K) \to K_*(C * K)$ is an isomorphism. Then the following diagram implies the splitting of A_X:

$$
\begin{array}{ccc}
K_*(X) & \xrightarrow{\ A_X\ } & K_*(C^*X) \\
\phi_* \downarrow & & \phi' \downarrow \\
K_*(K) & \xrightarrow{\ A_K\ } & K_*(C^*K) \\
\gamma_* \downarrow & & \\
K_*(X) & &
\end{array}
$$

$\square$

It is known [Ro1] that the monomorphism version of the coarse Baum-Connes conjecture implies the Gromov-Lawson conjecture. This together with Theorem 6.4 gives the following

Corollary 6.5. *The Gromov-Lawson conjecture holds for manifolds with asymptotically piecewise Euclidean universal covers.*

In view of the main result of the next section an affirmative answer to the following question would eliminate an approach to the Novikov Conjecture via hypersphericity (see a remark on page 35 of [G4]).

Problem 6.6. *Is every hypereuclidean (hyperspherical) manifold asymptotically piecewise Euclidean?*

§7 Expanders are not asymptotically piecewise Euclidean

Let X be a finite graph, we denote by V the set of vertices and by E the set of edges in X. We will identify the graph X with its set of vertices V. Every graph is a metric space with respect to the natural metric where every edge has the length one. For a subset $A \subset X$ we define the boundary $\partial A = \{x \in X \mid dist(x, A) = 1\}$. Let $|A|$ denote the cardinality of A.

DEFINITION [Lu]. An *expander* with a conductance number c and the degree d is an infinite sequence of finite graphs $\{X_n\}$ with the degree d such that $|X_n|$ tends to infinity and for every $A \subset X_n$ with $|A| \leq |X_n|/2$ there is the inequality $|\partial A| \geq c|A|$.

Let X be a finite graph, we denote by P all nonordered pairs of distinct points in X. For every nonconstant map $f : X \to l_2$ to the Hilbert space we introduce the number

$$D_f^2 = \frac{\frac{1}{|K|}\Sigma_{\{x,y\}\in P}\|f(x) - f(y)\|^2}{\frac{1}{|E|}\Sigma_{\{x,y\}\in E}\|f(x) - f(y)\|^2}.$$

If X is a graph with the degree d and with $|X| = n$, then $|P| = n(n-1)/2$ and $|E| = dn/2$.

The following Lemma is well-known. It can be derived from [M, Proposition 3]. It also can be obtain from the equality

$$\lambda_1(X) = \inf\{\frac{\|df\|^2}{\|f\|^2} \mid \Sigma f(x) = 0\}$$

for the first positive eigenvalue of the Laplacian on X and the Cheeger's inequality (see Proposition 4.2.3 in [Lu]).

Lemma 7.1. *Let $\{X_n\}$ be an expander. Then there is a constant c_0 such that $D^2_{f_n} \leq c_0$ for all n for all possible maps $f_n : X_n \to l_2$ to the Hilbert space.*

Corollary 7.2. *For every sequence of 1-Lipschitz maps $f_n : X_n \to l_2$ there is the inequality $\frac{1}{|P_n|}\Sigma_{P_n}\|f(x) - f(y)\|^2 \leq c_0$ for every n.*

Proof. In the case of 1-Lipschitz map we have

$$\frac{1}{|E|}\Sigma_{\{x,y\}\in E}\|f(x) - f(y)\|^2 \leq 1.$$

$\square$

We say that a metric space X contains an expander $\{X_n\}$ if there is a sequence of isometric embeddings $X_n \to X$.

Theorem 7.3. *An asymptotically piecewise Euclidean metric space X cannot contain an expander.*

Proof. Assume the contrary, i.e. assume that X contains an expander of degree d. We enumerate all the graphs in the expander by a subsequence of $\mathbf{N}$ in such a way that $|X_n| = n$. Let $x_0 \in X$ be a base point. Let $n(t) = \min\{n \mid X_n \cap \partial B_t(x_0) \neq \emptyset\}$. We note that $n(t)$ tends to infinity as t approaches infinity. We consider a proper function $f : X \to \mathbf{R}_+$ such that $f(x) < \log_d n(\|x\|)/4$. Then like in the proof of Lemma 3.1 one can take an 1-Lipschitz map $\phi : X \to N$ to an asymptotic polyhedron such that for every $r > 0$ there is a compact set $C_r \subset X$ with $diam(\phi^{-1}(B_r(y))) \leq f(z)$ for every $z \in \phi^{-1}(B_r(y)) \setminus C_r$. Since X is asymptotically Euclidean, we may assume that N is an Euclidean asymptotic polyhedron. By Lemma 6.3 there is a uniform embedding $g : N \to l_2$. By the definition, there is a monotone function $\rho : \mathbf{R}_+ \to \mathbf{R}_+$ tending to infinity such that

$$\rho(d_N(x', y')) \leq \|g(x') - g(y')\| \leq d_N(x', y')$$

for all $x', y' \in N$.

We take r such that $\rho(r) > 2\sqrt{c_0}$ and we take n sufficiently large. Denote by $P'_n \subset P_n$ the set of pairs $\{x, y\}$ in $X_n \setminus C_r$ with $d(x, y) \geq \log_d(n/4)$. Since the degree of X_n is d, we have

$$|B_k(v) \cap X_n| \leq 1 + d + d^2 + \cdots + d^k \leq 2d^k.$$

Therefore at least $n(n - 2\frac{n}{4})/2 = n^2/4$ such pairs are contained in X_n. Then for sufficiently large n we have at least $n^2/8$ such pairs in $X_n \setminus C_r$. Show that every such pair

$\{x, y\}$ in $X_n \setminus C_r$ satisfies the inequality $d_N(\phi(x), \phi(y)) > r$. Indeed, if we assume that $d_N(\phi(x), \phi(y)) \leq r$, then we will obtain that

$$diam\phi^{-1}(B_r(\phi(y))) \geq d(x, y) \geq \log_d(n/4).$$

On the other hand, we have that

$$diam\phi^{-1}(B_r(\phi(y))) \leq f(x) < \log_d(n(\|x\|)/4).$$

Since $n \geq n(\|x\|)$, we obtain the contradiction

$$diam\phi^{-1}(B_r(\phi(y))) < \log_d(n/4).$$

Let $f = g \circ \phi$. By the Corollary 7.2 we have

$$c_0 \geq \frac{1}{|P_n|}\Sigma_{P_n}\|f(x) - f(y)\|^2 \geq \frac{1}{|P_n|}\Sigma_{P_n'}\|f(x) - f(y)\|^2 \geq \frac{n^2}{8|P_n|} \min_{P_n'} \|f(x) - f(y)\|^2 =$$

$$\frac{n^2}{4n(n-1)}\|f(\tilde{x}) - f(\tilde{y})\|^2 \geq \frac{1}{4}\|f(\tilde{x}) - f(\tilde{y})\|^2 = \frac{1}{4}\|g(\phi(\tilde{x})) - g(\phi(\tilde{y}))\|^2 \geq$$

$$\frac{1}{4}\rho^2(d_N(\phi(\tilde{x}), \phi(\tilde{y}))) \geq \frac{1}{4}\rho^2(r) > c_0.$$

This contradiction completes the proof. $\square$

REMARK. One can show (see [M]) that for every $p \geq 1$ the number

$$D_f^p = \frac{\frac{1}{|P|}\Sigma_{\{x,y\}\in K}\|f(x) - f(y)\|^p}{\frac{1}{|E|}\Sigma_{\{x,y\}\in E}\|f(x) - f(y)\|^p} \leq c_0$$

is bounded from above by the same number c_0 for all maps $f : X \to l_p$ to the Banach space l_p. This implies in particular that an expander is not uniformly embeddable in l_p for any p.

Problem 7.4. *Given $p > 0$, does the Novikov conjecture holds for a group Γ which admits a uniform embedding in l_p ?*

It is known that the answer is 'yes' for $p = 2$ [Yu2] and for $p = 1$ [D-G-L-Y].

References

[B-D]   G. Bell and A. Dranishnikov, *On Asymptotic Dimension of Groups*, Algebraic and Geometric Topology **1** (2001), 57-71.

[C-G-M]   A. Connes, M. Gromov, H. Moscovici, *Group cohomology with Lipschitz control and higher signatures*, GAFA **3** (1993), 1-78.

[D]   M. Davis, *Groups generated by reflections and aspherical manifolds not cavered by Euclidean space*, Ann. Math. **(2) 117** (1983), 293-325.

[Dr1]   A. Dranishnikov, *Asymptotic topology*, Russian Math. Surveys **55:6** (2000), 71-116.

[Dr2]   A. Dranishnikov, *On large scale properties of manifolds*, Preprint math.GT/9912062 (1999).

[D-F]   A. Dranishnikov and S. Ferry, *On the Higson-Roe corona*, Russian Math. Surveys **52 no 5** (1997), 1017-1028.

[D-G-L-Y] A. Dranishnikov, G Gong, V. Lafforgue and G. Yu, *Uniform embedding into Hilbert space and a question of Gromov*, Canad. Math. Bull. (to appear).

[D-J]   A. Dranishnikov and T. Januszkiewicz, *Every Coxeter group acts amenably on a compact space*, Preprint math.GT/9911245, to appear in Topology Proceedings.

[F-H]   F.T. Farrell and W.-C. Hsiang, *On Novikov's conjecture for nonpositively curved manifolds, I*, Ann. Math. **113** (1981), 199-209.

[F-R-R]   S. Ferry, A. Ranicki and J. Rosenberg; Editors, *Novikov conjectures, index theorems and rigidity, Vol. 1, 2. London Math. Soc. Lecture Note Ser., 226,*, Cambridge Univ. Press, Cambridge, 1995.

[F-W]   S. Ferry and S. Weinberger, *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993), 147–163, London Math. Soc. Lecture Note Ser., 226,*, Cambridge Univ. Press, Cambridge, 1995.

[G1]   M. Gromov, *Asymptotic invariants of infinite groups*, Geometric Group Theory, vol 2, Cambridge University Press, 1993.

[G2]   M. Gromov, *Large Riemannian manifolds*, Lecture Notes in Math. **1201** (1985), 108-122.

[G3]   M. Gromov, *Positive curvature, macroscopic dimension, spectral gaps and higher signatures*, Functional Analysis on the eve of the 21st century, Vol. 2, Progr. Math. **132** (1996), 1-213..

[G4]   M. Gromov, *Spaces and questions*, Preprint (1999).

[G5]   M. Gromov, *Prospects in mathematics (Princeton, NJ, 1996)*, Amer. Math. Soc., Providence, RI, 1999, pp. 45–49,.

[G-L]   M. Gromov and H.B. Lawson, *Positive scalar curvature and the Dirac operator*, Publ. I.H.E.S. **58** (1983), 83-196.

[H-R1]   N. Higson and J. Roe, *The Baum-Connes conjecture in coarse geometry*, LMS Lecture Notes **227** (1995), 227-254.

[H-R2]   N. Higson and J. Roe, *Amenable action and the Novikov conjecture*, J. Reine Angew. Math. **519** (2000), 143-153.

[K-M]   J. Kaminker and J. Miller, *A comment on the Novikov conjecture*, Proc. Amer. Math. Soc. **83** (1981), 656-658.

[Kee]   J. Keesling, *The one-dimensional Čech cohomology of the Higson compactification and its corona*, Topology Proc. **19** (1994), 129–148.

[Lu]   A. Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Birkhauser, Basel-Boston-Berlin, 1994.

[M]   J. Matoušek, *On embedding expanders into l_p-spaces*, Israel J. Math. **102** (1997), 189-197.

[Ro1]   J. Roe, *Coarse cohomology and index theory for complete Riemannian manifolds*, Memoirs Amer. Math. Soc. No. 497, 1993.

[Ro2]   J. Roe, *Index theory, coarse geometry, and topology of manifolds*, CBMS Regional Conference Series in Mathematics, Number 90 (1996).

[Ros]   J. Rosenberg, *C^*-algebras, positive scalar curvature, and the Novikov conjecture*, Publ. I.H.E.S. **58** (1983), 197-212.

[Sit]   K. Sitnikov, *The duality law for non-closed sets*, (Russian) Doklady Akad. Nauk SSSR. (N.S.) **81** (1951), 359–362.

[St]   N. Steenrod, *Regular cycles of compact metric spaces*, Ann. Math. **41** (1940), 833-851.

[S-W]   J. Siegel and F. Williams, *Uniform bounds for isoperimetric problems*, Proc. Amer. Math. Soc. **107** (1989), no 2, 459–464.

[Yu1]   G. Yu, *The Novikov conjecture for groups with finite asymptotic dimension*, Ann. of Math **147** (1998), no. 2, 325-355.

[Yu2]   G. Yu, *The coarse Baum-Connes conjecture for groups which admit a uniform embedding into Hilbert space*, Inventiones Mathematicae **139** (2000), no 1, 2001-240.

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